Building a Smooth Yield Curve

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Preliminaries

As before, we will use continuously compounding Act/365 rates for both the zero coupon rates and forward rates. Time is measured in years from now. Recall the following quantities:

<table>
<thead>
<tr>
<th>Name</th>
<th>Symbol</th>
<th>Formulas</th>
</tr>
</thead>
<tbody>
<tr>
<td>discount factor</td>
<td>$P(t)$</td>
<td>$e^{-r(t) \cdot t}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$e^{-\int_0^t f(s) ds}$</td>
</tr>
<tr>
<td>forward discount factor</td>
<td>$P(t,T)$</td>
<td>$e^{-f(t,T)(T-t)}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{P(T)}{P(t)}$</td>
</tr>
<tr>
<td>zero rate</td>
<td>$r(t)$</td>
<td>$-\frac{1}{t} \log P(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{1}{t} \int_0^t f(s) ds$</td>
</tr>
<tr>
<td>discrete forward rate</td>
<td>$f(t,T)$</td>
<td>$\frac{r(T) \cdot T - r(t) \cdot t}{T - t}$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$-\frac{1}{T - t} \log P(t, T)$</td>
</tr>
<tr>
<td>instantaneous forward rate</td>
<td>$f(t)$</td>
<td>$-\frac{d}{dt} \log P(t)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\lim_{\tau \downarrow 0} f(t, t + \tau)$</td>
</tr>
<tr>
<td></td>
<td></td>
<td>$\frac{d}{dt} (r(t) \cdot t)$</td>
</tr>
</tbody>
</table>
Mechanics

Let

\[ 0 = t_0 < t_1 < \cdots < t_n \]

be an ordered set of times. A yield curve is represented by any of the following:

- **Discount factors** $P(t_1), P(t_2), \ldots, P(t_n)$,
- **Zero rates** $r(t_1), r(t_2), \ldots, r(t_n)$, or
- **Forward rates** $f(t_0, t_1), f(t_1, t_2), \ldots, f(t_{n-1}, t_n)$.

These are all equivalent, as can be seen by the relations

\[
\begin{align*}
P(t_i) &= e^{-r(t_i) \cdot t_i}, \\
r(t_i) \cdot t_i &= r(t_{i-1}) \cdot t_{i-1} + f(t_{i-1}, t_i) (t_i - t_{i-1}).
\end{align*}
\]
What is Smoothness?

Smoothness is a measure of the degree to which a function wriggles. A smoothness penalty is often defined as

$$\int (f''(t))^2 \, dt$$

or

$$\int |f'(t)| \, dt.$$ 

The zero rates are averages of forward rates

$$r(t) = \frac{1}{t} \int_0^t f(s) \, ds$$

and are therefore smoother. This implies that if we smooth the forward rates, the zero rates will also be smooth. So what will be the criterion of smoothness for our discrete sequence of forward rates?

$$f_1 = f(t_0, t_1), f_2 = f(t_1, t_2), \ldots, f_n = f(t_{n-1}, t_n)$$
We shall choose a discrete version of one of the continuous time measures we have already seen. One possibility is that a forward rate should be “close” to its neighbors, e.g.

\[ \sum_{i=2}^{n} |f_i - f_{i-1}| \]

should be small. The absolute value is tricky to deal with, so we will use the squares of the differences instead

\[ S(f) = \sum_{i=2}^{n} (f_i - f_{i-1})^2. \]

Another possibility is

\[ \sum_{i=2}^{n-1} \left( f_{i-1} - 2f_i + f_{i+1} \right)^2. \]
Why is Smoothing Possible?

The prices of the traded input instruments (deposits, Eurodollar futures, and swaps) are functions of the forward rates. The forward rates are not uniquely determined by these prices. There are always extra degrees of freedom. When we bootstrapped the LIBOR curve in the previous lecture, we chose a constant forward rate interpolation method in order to specify the curve.

We used a total of 23 constraints (instruments) to construct the curve. The number of unknowns (forward rates) is equal to the number of knot times. A knot time is a time whose corresponding discount factor impacts the value of any of the input instruments:
• Deposit begin and end dates

• Futures IMM dates and end dates

• Swap fixed cashflow payment dates

We have at least 41 knot times and only 23 constraints. That leaves at least an extra 18 degrees of freedom! We will use them for smoothing.
The Dangers of Overfitting

*Underfitting* is when the number of constraints is greater than the number of free variables. This is relatively benign. Generally, not every constraint can be satisfied. Fewer constraints than free variables results in *overfitting*. Ad-hoc procedures for fixing extra variables usually result in unpleasant consequences. Extra variables go haywire. Overfitting is extremely widespread but usually hidden! Watch out for it in

- Curve building
- Model calibration
- Market (BGM) models
- Everywhere else
Overfitted models are unstable, have little explanatory and predictive power, yet provide a false sense of security (because all inputs are matched).

Ways to deal with overfitting:

• Recognize it

• Avoid it

• Formulate a goal that extra parameters are designed to achieve
A Simple Example

Consider two instruments

- A Eurodollar deposit
- A Eurodollar future

Assume that the deposit settles at time $t_0$ and pays at time $t_2$. The Eurodollar covers the time period from $t_1$ to $t_3$. The times are arranged in order $t_0 < t_1 < t_2 < t_3$. Assume all day bases are Act/365. Define

$$
\tau_1 = t_1 - t_0, \\
\tau_2 = t_2 - t_1, \\
\tau_3 = t_3 - t_1.
$$
The three forwards involved are

\[ f_1 = f(t_0, t_1), \]
\[ f_2 = f(t_1, t_2), \]
\[ f_3 = f(t_2, t_3). \]

Two equations must be satisfied by the forward rates,

\[ D = \exp(f_1 \tau_1 + f_2 \tau_2) - 1 \]
\[ F = \exp(f_2 \tau_2 + f_3 \tau_3) - 1 \]

where \( D \) is the deposit rate and \( F \) is the rate implied by the futures price. Expressed in term of forward rates, the constraints become

\[ \tau_1 f_1 + \tau_2 f_2 = \log(D(\tau_1 + \tau_2) + 1) \]
\[ \tau_2 f_2 + \tau_3 f_3 = \log(F(\tau_2 + \tau_3) + 1). \]

We have two equations and three unknowns. We want the smoothest curve possible. Recall our definition of smoothness. The goal is to minimize

\[ S(f_1, f_2, f_3) = (f_2 - f_1)^2 + (f_3 - f_2)^2 \]
using the extra degree of freedom. By direct substitution

\[
\begin{align*}
    f_1 &= \log \left( D \left( \tau_1 + \tau_2 \right) + 1 \right) - f_2 \tau_2 \\
    f_3 &= \log \left( F \left( \tau_2 + \tau_3 \right) + 1 \right) - f_2 \tau_3
\end{align*}
\]

Plugging into \( S(f_1, f_2, f_3) \) leaves a one dimensional unconstrained optimization problem. Find \( f_2 \) where the minimum of

\[
S(f_2) = \left( \left( 1 + \frac{\tau_2}{\tau_1} \right) f_2 - \frac{\log \left( D \left( \tau_1 + \tau_2 \right) + 1 \right)}{\tau_1} \right)^2 + \left( \left( 1 + \frac{\tau_2}{\tau_3} \right) f_2 - \frac{\log \left( F \left( \tau_2 + \tau_3 \right) + 1 \right)}{\tau_3} \right)^2
\]

is attained.

**Do this as an exercise!**
The Method in Detail

Step 1: Identify the instruments used in constructing the curve

- Deposits / LIBOR rates
- Eurodollar futures
- Swaps

We will call them “curve instruments”. 
Step 2: Identify knot times

The knot times are characterized as the set of times whose corresponding discount factors impact the value of the curve instruments. Knot times include

- Today
- Deposit accrual begin dates and end dates
- Eurodollar futures underlying forward LIBOR begin and end dates
- Swap settlement dates and fixed rate payment dates

Sort the knot times

\[ 0 = t_0 < t_1 < \cdots < t_n, \]

breaking the timeline into non-overlapping periods.
Step 3: Assign a forward rate (free variable) to each time period

\[ f_1 = f(t_0, t_1), f_2 = f(t_1, t_2), \ldots, f_n = f(t_{n-1}, t_n), \]

establishing \( n \) independent variables.

Step 4: Express the curve instrument prices as functions of the forward rates

- First express the curve instrument prices through the discount factors at the knot times

\[ P(t_1), P(t_2), \ldots, P(t_n). \]

This is always possible by definition of the knot times.

- Second express the discount factors in terms of the forward rates. This can be done recursively

\[
\begin{align*}
P(t_0) &= 1, \\
P(t_i) &= P(t_{i-1}) \exp (-f_i \cdot (t_i - t_{i-1})).
\end{align*}
\]
Step 5: Define constraints

The smoothed curve should imply curve instrument prices which are a close match for market quotes. In other words

\[ V_i(f_1, f_2, \ldots, f_n) = v_i, \quad i = 1, 2, \ldots, m. \]

Here \( V_1(f), V_2(f), \ldots, V_m(f) \) are the prices as a function of the forward rates, and \( v_1, v_2, \ldots v_m \) are the market quotes we need to match. If \( m = n \) we are done; there are no extra degrees of freedom to use for smoothness. If \( m > n \) we have too many constraints; we cannot match all the prices. If, as is the usual case, \( m < n \) we can use the extra degrees of freedom for smoothing.
Step 6: Formulate a smoothness criteria

Use

\[ S(f) = \sum_{i=2}^{n} (f_i - f_{i-1})^2 , \]
\[ S(f) = \sum_{i=2}^{n-1} \left( f_{i-1} - 2f_i + f_{i+1} \right)^2 , \]

or something else.
Step 7: Solve a non-linear constrained optimization problem

Mathematically we must find $f$ such that $S(f)$ is a minimum among all possible solutions $V(f) = \nu$. This is a very hard problem. It is a multidimensional (50+) minimization with non-linear constraints. This pushes the limit of numerical techniques.

An alternative is to get rid of the hard constraints. Use soft constraints instead. Replace $V(f) = \nu$ with the mispricing objective function

$$M(f) = \sum_{i=1}^{n} (V_i(f) - v_i)^2.$$ 

The closer the fit to market prices, the smaller the mispricing $M(f)$. The reformulated problem is to find $f$ such that

$$\Omega(f; w) = wS(f) + M(f)$$
attains its minimum. We have included a control variable $w$ to specify the relative importance of smoothness versus mispricing. This is a much simpler unconstrained optimization problem. Even Excel can handle it! The curve will no longer exactly price the input instruments. This is not necessarily bad – there is always a bid/ask spread. We can fall within the bid/ask spread by choosing the right value for $w$. Typical bid/ask spreads are

- Deposits: $\frac{1}{2}$bp
- Eurodollars: $\frac{1}{4}-\frac{1}{2}$bp
- Swaps: 1-2bp
Advanced: Connecting the Dots

We know the discrete forward rates for the knot times. But, how do we value other instruments that do not necessarily “hit” the knot times? Interpolate.

Assuming flat forward rates between the knot times is the easy way out. We have worked hard to make the curve “smooth”. Why not connect the dots smoothly? There are several choices for interpolation:

- Polynomial
- Polynomial spline
- Exponential spline
- Other families of smooth functions
How do we choose? Be consistent.

For instance, if you chose

\[ S(f) = \sum_{i=2}^{n-1} (f_{i-1} - 2f_i + f_{i+1})^2 \]

as your smoothness constraint, then why not try to extend this by minimizing \( \int (f''(t))^2 \, dt \)? In other words, find a twice differentiable function \( f(\cdot) \) such that

\[ \int_0^{tn} (f''(s))^2 \, ds \]  

(1)

is a minimum among all possible solutions

\[ \frac{1}{t_i - t_{i-1}} \int_{t_{i-1}}^{t_i} f(s) \, ds = f_i, \quad i = 1, 2, \ldots n. \]  

(2)
In order to achieve this we will invoke the following theorem*

**Theorem.** The term structure of instantaneous forward rates \( f(t) \), \( 0 \leq t \leq T \), that minimizes the smoothness criteria (1) subject to the constraints (2) is a forth-order polynomial spline with knot times \( \{t_i\}_{i=0}^n \).

The coefficients of the spline are determined by solving a linear system of equations.

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Final Words of Wisdom

- It is not always appropriate to smooth the yield curve*.

- If the market data used to construct the curve is bad, no amount of smoothing will fix it.

- If different curve instruments have incompatible prices (e.g. on-the-run and off-the-run Treasuries), smoothing will make matters worse.

- When building a curve that has a scarce set of curve instruments (e.g. Treasury curve), some extra inputs should usually be used to “get it right”†.

*For instance, when calculating price sensitivities to bucketed discrete forward rates, yield curve shocks will bleed into other buckets.

†E.g., Treasury curve shape between 10 and 30 years forced to match the shape of the LIBOR curve.